

# ON MODULAR FUNCTIONS WITH A CUSPIDAL DIVISOR

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## Abstract

The aim of this paper is the generalization of the following equivalence due to Kohnen (in [6]): Let  $f$  be a modular function of integral weight with respect to  $\Gamma_0(N)$ ,  $N$  square-free. Then  $f$  has a cuspidal divisor (i.e. zeros and poles supported at the cusps) if and only if  $f$  is an  $\eta$ -quotient. We first present Theorem 3, which states that Kohnen's result still holds when  $N = 4M$  and  $8M$ , with  $M$  an odd and square-free integer, and then Theorem 4, which extends the equivalence to the cases

- $N = 9M$  and  $N = 27M$ , with  $M$  a square-free integer coprime to 3,
- $N = 16M$  and  $N = 32M$ , with  $M$  an odd and square-free integer,
- $N = 25M$  and  $N = 125M$ , with  $M$  a square-free integer coprime to 5,

by introducing a generalization of the classical Dedekind  $\eta$ -function.

## I Introduction to the main result.

Let  $f$  be a meromorphic modular function of integral weight  $k$  on a congruence subgroup  $\Gamma$  of finite index in  $\Gamma(1) := \mathrm{SL}_2(\mathbf{Z})$ , i.e. it is a meromorphic function on the complex upper half-plane  $\mathcal{H}$ , satisfies the usual transformation in weight  $k$  under the action of  $\Gamma$ , and is meromorphic at the cusps. In the present paper we shall try to characterize  $f$  when it has no zeros or poles on  $\mathcal{H}$ . In the context of Riemann surfaces, we usually say that the function has a cuspidal divisor (in the sense that zeros and poles are supported at the cusps). When  $\Gamma = \Gamma(1)$ , one concludes easily from the valence formula that  $12|k$  and  $f$  is proportional to  $\Delta^{k/12}$  where  $\Delta$  is the modular discriminant. In 2003, Winfried Kohnen extended this well-known result to the case  $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), N|c \right\}$  where  $N$  is a square-free integer (see [6], Thm. 2): he showed that  $f$  was an  $\eta$ -quotient of level  $N$  in the following sense: given  $f_1, f_2, \dots, f_d$ , complex-valued functions from the complex upper half-plane, a function  $f$  from  $\mathcal{H}$  to  $\mathbf{C}$  is a  $f_1, f_2, \dots, f_d$ -quotient of level  $N$  if there exist complex numbers  $c$  and  $(a_{t,i})_{t|N, 1 \leq i \leq d}$  such that  $f = c \prod_{t|N} (f_1|V_t)^{a_{t,1}} \cdots (f_d|V_t)^{a_{t,d}}$  where we set  $(f_i|V_t)(\tau) = f_i(t\tau)$  (for all  $\tau \in \mathcal{H}$ ) and where the complex powers are defined by the principal branch of the complex logarithm. As usual,  $\eta$  denotes the Dedekind function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (q = e^{2i\pi\tau}).$$

In Kohnen's paper, following Thm. 2, one can read:

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We think that the assertion of Theorem 2 or a slightly weaker statement eventually would be true for arbitrary  $N$ , but it seems that this cannot be proved by the method employed here.

Surprisingly, this Theorem 2 appears to fail for some smaller congruence subgroups but via a generalization of the Dedekind  $\eta$  function this result can be extended. For  $\chi$  a Dirichlet character modulo  $N$  ( $N > 1$ ), we will refer to the following function:

$$\eta_\chi(\tau) := \prod_{n=1}^{\infty} (1 - \zeta_u q^n)^{\bar{\chi}(n)} (1 - \zeta_u^2 q^n)^{\bar{\chi}(2n)} \dots (1 - \zeta_u^u q^n)^{\bar{\chi}(un)} \quad (\tau \in \mathcal{H})$$

where  $\zeta_u := e^{2i\pi/u}$ , as the Generalized Dedekind  $\eta$  function attached to  $\chi$ . Denoting  $\mathbf{1}_N$  the principal character modulo  $N$ , we convince that  $\eta_{\mathbf{1}_1}$  is the classical Dedekind function. This convention will make sense in the next section. We shall first explain why they are natural generalizations of  $\eta$  and show that  $\eta_\chi$  satisfies a particular modular transformation under the action of a Hecke congruence subgroup. More precisely, we prove:

**Theorem 1.** *Let  $\chi$  be a real Dirichlet character (i.e. taking its values in  $\{-1, 1\}$ ) with modulus  $N > 1$  and conductor  $u$ . Let  $Q := u^2 \prod_{p|N, p \nmid u} p$ . Then  $\eta_\chi$  is a Parabolic Generalized Modular Function (of weight 0) on  $\Gamma_0(Q)$  with a cuspidal divisor. Further,  $\eta_\chi$  is an  $\eta$ -quotient if and only if  $\chi$  is principal.*

As it is less well-known, we shall recall the definition of a Generalized Modular Function in the next section. The situation appears to be more complicated when  $\chi$  is non real. Indeed, when  $\chi$  takes non real values, the function  $\eta_\chi$  is not a Generalized Modular Function anymore. By now, even when  $\chi$  is real, it is not clear whether  $\eta_\chi$  is a classical meromorphic modular function with a cuspidal divisor for all modulus  $N$ . Nevertheless we have the following statement:

**Theorem 2.** *For an odd prime number  $p$ , let  $(\cdot/p)$  denotes the Legendre symbol attached to  $p$  and let  $\psi$  denotes the non principal (real) Dirichlet character mod 4. Then  $\eta_{(\frac{\cdot}{3})}$ ,  $\eta_\psi$  and  $\eta_{(\frac{\cdot}{5})}$  are Generalized Modular Functions with unitary characters, taking values in the twelfth roots of unity, on the respective groups  $\Gamma_0(9)$ ,  $\Gamma_0(16)$  and  $\Gamma_0(25)$  with a cuspidal divisor and which are not  $\eta$ -quotient.*

Equivalently, the twelfth powers of  $\eta_{(\frac{\cdot}{3})}$ ,  $\eta_\psi$  and  $\eta_{(\frac{\cdot}{5})}$  are meromorphic modular functions with a cuspidal divisor. Namely, those functions are:

$$\eta_{(\frac{\cdot}{3})}(\tau) = \prod_{n=1}^{\infty} \left( \frac{1 - e^{2i\pi/3} q^n}{1 - e^{4i\pi/3} q^n} \right)^{(\frac{n}{3})}, \quad \eta_\psi(\tau) = \prod_{n=1}^{\infty} \left( \frac{1 - iq^n}{1 + iq^n} \right)^{\psi(n)},$$

$$\eta_{(\frac{\cdot}{5})}(\tau) := \prod_{n=1}^{\infty} \left( \frac{1 - \frac{1-\sqrt{5}}{2} q^n + q^{2n}}{1 - \frac{1+\sqrt{5}}{2} q^n + q^{2n}} \right)^{(\frac{n}{5})} \quad (\tau \in \mathcal{H}).$$

After proving those theorems, we will essentially introduce two results. The first one is a extension of Kohnen's theorem:

**Theorem 3.** *Let  $N$  be an integer of the form  $4M$  or  $8M$  where  $M$  is odd and square-free. Any meromorphic modular function of integral weight on  $\Gamma_0(N)$  is an  $\eta$ -quotient of level  $N$ .*

The second one partially answers to Kohnen's expectations:

**Theorem 4.** *Let  $\chi_1$  and  $\chi_2$  be the two primitive Dirichlet characters modulo 5 distinct from  $(\cdot/5)$ . Any meromorphic modular function of integral weight on  $\Gamma_0(N)$  with a cuspidal divisor can be expressed as*

- (i) an  $\eta, \eta(\frac{\cdot}{3})$ -quotient of level  $N$  when  $N$  is of the form  $9M$  or  $27M$  with  $M$  square-free and coprime with 3,
- (ii) an  $\eta, \eta_\psi$ -quotient of level  $N$  when  $N$  is of the form  $16M$  or  $32M$  with  $M$  odd and square-free,
- (iii) an  $\eta, \eta(\frac{\cdot}{5}), \eta_{\chi_1}, \eta_{\chi_2}$ -quotient of level  $N$  when  $N$  is of the form  $25M$  or  $125M$  with  $M$  square-free and coprime with 5.

We deeply think that the method employed here can be extended to prove some Thm. 4-like results for other congruence subgroups and hopefully finish the classification for arbitrary  $N$ .

## II Early results and notations.

The proofs of those four theorems use extensively the basic ingredients in the theory of Generalized Modular Functions (usually referred as GMFs) introduced by Marvin Knopp and Geoffrey Mason in [5]. Let us briefly recall the story of GMFs. Let  $\Gamma \subset \Gamma(1)$  be a subgroup of finite index. A *Generalized Modular Function* (of weight 0) on  $\Gamma$  is a holomorphic function  $f$  defined on  $\mathcal{H}$  such that

- (i)  $(f|_0\gamma) = \nu(\gamma)f$  for all  $\gamma \in \Gamma$  (where  $|_0$  denotes the usual slash operator of weight 0) and where  $\nu : \Gamma \rightarrow \mathbf{C}^*$  is a (not necessarily unitary) character of  $\Gamma$ .
- (ii)  $f$  is meromorphic at the cusps of  $\Gamma$ .

Further, if  $f$  satisfies the following additional condition

- (iii)  $\nu(\gamma) = 1$  for every element  $\gamma \in \Gamma$  of trace 2.

then we call  $f$  a *parabolic GMF* (PGMF) with multiplier system  $\nu$  (of weight 0).

One main result of Knopp and Mason can be stated as follow (see [5], Thm. 2 for a general version): if  $f$  is a PGMF on  $\Gamma$  with a cuspidal divisor (i.e. zeros and poles supported at the cusps), then its  $q$ -logarithmic derivative

$$g := \frac{\theta f}{f} = \frac{1}{2i\pi} \frac{f'}{f} \quad (1)$$

is a classical modular form of weight 2 on  $\Gamma$  with trivial character and satisfies

- (a) The constant term in the expansion of  $g$  at every cusps  $s = a/c$  of  $\Gamma$  takes the form  $l_s/(w_s c^2)$  where  $l_s \in \mathbf{Z}$  (namely  $\text{ord}_s(f)$ ) and  $w_s$  is the width of  $s$ .

Conversely, if  $g$  is a classical modular form of weight 2 on  $\Gamma$  with trivial character and satisfies condition (a), there is a PGMF  $f$  with a cuspidal divisor (uniquely determined up to multiplication with non-zero complex numbers) which satisfies Eq. (1).

We easily derive from this that any PGMF  $f$  with unitary multiplier system whose  $q$ -logarithmic derivative is a cusp form must be constant. Indeed,  $|f|$  is bounded on  $\mathcal{H}$  and therefore constant by elementary complex analysis.

There is also a helpful remark made by Knopp and Mason: if a subgroup of finite index  $\Gamma$  is generated by only parabolic and elliptic elements, then a parabolic multiplier system  $\nu$  (i.e. attached to a PGMF) must have finite order: by definition  $\nu(P) = 1$  for parabolic elements  $P$  and since elliptic elements  $E$  (of  $\Gamma(1)$ ) have finite order dividing 12,  $\nu(E)$  is a twelfth root of unity. We conclude that  $\nu$  has finite order dividing 12 from the fact that  $\Gamma$  is finitely generated. This situation happens when  $\Gamma$  is the group  $\Gamma_0(9)$ ,  $\Gamma_0(16)$  and  $\Gamma_0(25)$  since those groups have genus 0.

We also need to agree on some notations (we will try to use the same of those in the famous textbook of Diamond and Shurman [1]).

- The vector-valued Eisenstein series of weight 2 modulo  $N$  ( $\bar{v} = \overline{(c_v, d_v)} \in (\mathbf{Z}/N\mathbf{Z})^2$ )

$$G_2^{\bar{v}}(\tau) := \delta(\bar{c}_v) \zeta^{\bar{d}_v}(2) - \frac{4\pi^2}{N^2} \sum_{n=1}^{\infty} \sigma_1^{\bar{v}}(n) q_N^n$$

where

$$\delta(\bar{c}_v) := \begin{cases} 1 & \text{if } c_v \equiv 0 \pmod{N}, \\ 0 & \text{otherwise.} \end{cases}, \quad \zeta^{\bar{d}_v}(2) := \sum_{\substack{m \equiv d_v \pmod{N} \\ m \neq 0}} \frac{1}{m^2},$$

$$\sigma_1^{\bar{v}}(n) := \sum_{\substack{d|n \\ n/d \equiv c_v \pmod{N}}} |d| e^{2i\pi \frac{d_v d}{N}},$$

where the two sums are taken over positive and negative values of  $m$  and  $d$  and where the overline denotes the reduction modulo  $N$ .

- For  $\psi$  and  $\varphi$  two Dirichlet characters modulo  $u$  and  $v$ ,  $N = uv$ , we define

$$G_2^{\psi, \varphi} := \sum_{\substack{0 \leq c, e < u \\ 0 \leq d < v}} \psi(c) \bar{\varphi}(d) G_2^{\overline{(cv, d+ev)}}.$$

- Again, for  $\psi$  and  $\varphi$  two Dirichlet characters modulo  $u$  and  $v$ ,  $N = uv$ ,

$$E_2^{\psi, \varphi} := \delta(\psi) L(-1, \varphi) + 2 \sum_{n=1}^{\infty} \sigma_1^{\psi, \varphi}(n) q^n$$

where

$$\sigma_1^{\psi, \varphi}(n) := \sum_{d|n} \psi\left(\frac{n}{d}\right) \varphi(d) d$$

(where  $\delta(\psi) = 1$  when  $\psi$  is principal, 0 otherwise, and  $L(-1, \varphi)$  denotes the value of the Dirichlet  $L$ -function attached to  $\varphi$  at  $-1$ ). Further, we have  $G_2^{\psi, \varphi} = (-4\pi^2 g(\bar{\varphi})/v^2) E_2^{\psi, \varphi}$  which is a modular form of weight 2 for  $\Gamma_0(N)$  attached to the Dirichlet character  $\psi\varphi$  (where  $g(\bar{\varphi})$  denotes the Gauss sum of  $\bar{\varphi}$ , again see [1]).

### III Brief study of Generalized $\eta$ functions and proofs of Thm 1 and 2.

In the scope of proving Thm. 1, we first show that  $\eta_\chi$ , when  $\chi$  is primitive with modulus  $u$ , is well-defined and that its  $q$ -logarithmic derivative is a particular Eisenstein series of weight 2 with trivial character on  $\Gamma_0(u^2)$ .

**Lemma 1.** *Let  $\chi$  be a primitive Dirichlet character of conductor  $u > 1$ . Then  $\eta_\chi$  is holomorphic, has no zeros on  $\mathcal{H}$ , and satisfies  $\theta\eta_\chi/\eta_\chi = -(g(\bar{\chi})/2)E_2^{\chi, \bar{\chi}}$ .*

Since  $\theta\eta/\eta = -(1/2)E_2^{1, \bar{1}}$ , Lemma 1 makes the convention  $\eta_{1_1} := \eta$  clear. We sketch the proof.

*Proof.* Since the series

$$S(\tau) = \sum_{n=1}^{\infty} \sum_{\substack{1 \leq a \leq u \\ \gcd(a, u)=1}} \bar{\chi}(an) \log(1 - \zeta_u^a q^n)$$

converges absolutely uniformly on compact subsets of  $\mathcal{H}$  and from the calculation  $S = \theta\eta_\chi/\eta_\chi$ ,  $\eta_\chi$  is holomorphic and has no zeros and poles on  $\mathcal{H}$ , as the exponential of a holomorphic function on  $\mathcal{H}$ , and satisfies the logarithmic differentiation formula:

$$\begin{aligned} \frac{\theta\eta_\chi}{\eta_\chi}(\tau) &= - \sum_{n=1}^{\infty} \sum_{\substack{1 \leq a \leq u \\ \gcd(a,u)=1}} n\bar{\chi}(an) \frac{\zeta_u^a q^n}{1 - \zeta_u^a q^n} \\ &= - \sum_{\substack{1 \leq a \leq u \\ \gcd(a,u)=1}} \bar{\chi}(a) \sum_{n=1}^{\infty} n\bar{\chi}(n) \sum_{r=1}^{\infty} \zeta_u^{ar} q^{nr}. \end{aligned}$$

The absolute convergence of the series allows us to write

$$\begin{aligned} \frac{\theta\eta_\chi}{\eta_\chi}(\tau) &= - \sum_{\substack{1 \leq a \leq u \\ \gcd(a,u)=1}} \bar{\chi}(a) \sum_{m=1}^{\infty} \left( \sum_{d|m} d\bar{\chi}(d) \zeta_u^{am/d} \right) q^m \\ &= - \sum_{m=1}^{\infty} \left( \sum_{d|m} d\bar{\chi}(d) g\left(\frac{m}{d}, \bar{\chi}\right) \right) q^m, \end{aligned}$$

where  $g$  is the Gauss sum

$$g(n, \chi) = \sum_{\substack{1 \leq a \leq u \\ \gcd(a,u)=1}} \chi(a) \zeta_u^{an}.$$

But, since  $\bar{\chi}$  is primitive, its Gauss sum is separable for every  $n$ , i.e.  $g(n, \bar{\chi}) = \chi(n)g(\bar{\chi})$ . This gives

$$\frac{\theta\eta_\chi}{\eta_\chi}(\tau) = -g(\bar{\chi}) \sum_{m=1}^{\infty} \sigma_1^{\chi, \bar{\chi}}(m) q^m = -\frac{g(\bar{\chi})}{2} E_2^{\chi, \bar{\chi}}(\tau).$$

□

The second step in the proof of Thm. 1 is to show that we can restrain our study to the case where  $\chi$  is a primitive character. This is the following lemma:

**Lemma 2.** *Let  $\chi$  be a real Dirichlet character with modulus  $N > 1$  and conductor  $u$ . Write  $\chi = \chi_0 \mathbf{1}_N$  where  $\chi_0$  is a primitive Dirichlet character of conductor  $u$ . Then,*

$$\eta_\chi = \prod_{t|N} (\eta_{\chi_0}|V_t)^{\chi_0(t)\mu(t)}$$

where  $\mu$  denotes the Möbius function.

The proof is straightforward, as an application of Möbius transformation formula, and thus left to the reader (remember that  $\chi_0$  is real so there is no issue with the complex logarithm).

From now on we will assume that  $\chi$  is primitive. To determine whether  $\eta_\chi$  is a PGMF, and in the scope of condition (a), we have to find the constant term of  $\theta\eta_\chi/\eta_\chi$  at any cusp and apply Knopp and Mason's theorem.

**Lemma 3.** *For an integer  $u > 1$ , let  $\chi$  be a primitive Dirichlet character with conductor  $u$ . The constant term of  $\theta\eta_\chi/\eta_\chi$  at every cusps of the form  $k/u$ , where  $\gcd(k, u) = 1$ , is given by*

$$\chi(k) \left(\frac{u}{2\pi}\right)^2 L(2, \chi^2).$$

The function  $\theta\eta_\chi/\eta_\chi$  vanishes at the other cusps.

*Proof.* According to Lemma 1 it is enough to study  $E_2^{\chi, \bar{\chi}}$  at cusps. It turns out to be easier to calculate the constant term of  $G_2^{\chi, \bar{\chi}}$  and to use the relation  $-g(\chi)E_2^{\chi, \bar{\chi}} = (u/2\pi)^2 G_2^{\chi, \bar{\chi}}$ . It is not difficult to see that for any vector  $\bar{v} = \overline{(c_v, d_v)} \in (\mathbf{Z}/u^2\mathbf{Z})^2$ , the corresponding vector-valued Eisenstein series of weight 2 modulo  $u^2$ , namely  $G_2^{\bar{v}}$ , behaves like a quasi-modular form of weight 2 with respect to

$$\Gamma(u^2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I_2 \pmod{u^2} \right\}.$$

More precisely, we have the general transformation property

$$\forall \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma(1), \quad (G_2^{\bar{v}}|_2 \gamma)(\tau) = G_2^{\overline{v\gamma}}(\tau) - \frac{2i\pi c_\gamma}{u^4(c_\gamma \tau + d_\gamma)}$$

where  $|_2$  denotes the usual slash operator of weight 2. We know that  $G_2^{\chi, \bar{\chi}}$  is a modular form with respect to  $\Gamma_0(u^2)$  (remember that neither  $\chi$  or  $\bar{\chi}$  is principal). We thus have the following transformation property:

$$(G_2^{\chi, \bar{\chi}}|_2 \gamma) = \sum_{0 \leq c, d, e < u} \chi(c)\chi(d) \overline{G_2^{(cu, d+eu)\gamma}}, \text{ since } \sum_{0 \leq c, d < u} \chi(c)\chi(d) = 0.$$

For a cusp of the form  $k/s$ ,  $\gcd(k, s) = 1$ , one must compute  $\lim_{\text{Im}(\tau) \rightarrow \infty} (G_2^{\chi, \bar{\chi}}|_2 \gamma)(\tau)$  to obtain the constant term at this cusp, where  $\gamma = \begin{pmatrix} k & b \\ s & k' \end{pmatrix} \in \Gamma(1)$  maps the cusp  $\infty$  to the cusp  $k/s$  (such a  $(b, k')$  exists by Bézout's identity). Since

$$\begin{aligned} \overline{(cu, d+eu)\gamma} &= \overline{(cuk + sd + seu, bcu + k'd + k'eu)} \\ &= \overline{(u(se + ck) + sd, u(bc + k'e) + k'd)} \end{aligned}$$

as  $\text{Im}(\tau)$  tends to  $\infty$ , only the constant term remains:

$$\sum_{0 \leq c, d, e < u} \chi(c)\chi(d)\delta(u(se + ck) + sd)\zeta^{\overline{(u(bc + k'e) + k'd)}}(2).$$

If  $u \nmid s$  it sums to zero ( $d$  can be assumed to be relatively prime to  $u$  thanks to the term  $\chi(d)$ ). In the following, we assume that  $u|s$  and we will write  $s = tu$ . One remarks that the sum is not modified if we replace  $e$  or  $c$  by any of their representant modulo  $u$ : we can thus make the substitution  $(c, e) \begin{pmatrix} k & b \\ tu & k' \end{pmatrix} \mapsto (c, e)$  in the sum and we get

$$\sum_{0 \leq c, d, e < u} \chi(ck' - tue)\chi(d)\delta(u(c + td))\zeta^{\overline{(ue + k'd)}}(2).$$

The summand is zero unless  $c \equiv -td \pmod{u}$ : one obtains

$$\begin{aligned} \chi(-t) \sum_{0 \leq d, e < u} \chi(dk' + ue)\chi(d)\zeta^{\overline{(ue + k'd)}}(2) \\ = \chi(-t) \sum_{0 \leq d < u} \chi(d) \sum_{e=0}^{u-1} \left( \sum_{\substack{m \equiv ue + k'd \pmod{u^2} \\ m \neq 0}} \frac{\chi(m)}{m^2} \right) \end{aligned}$$

and calculating first the sum over  $0 \leq e < u$ , it gives

$$\chi(-t) \sum_{0 \leq d < u} \chi(d) \left( \sum_{\substack{m \equiv k'd \pmod{u} \\ m \neq 0}} \frac{\chi(m)}{m^2} \right).$$

Now remark that  $k'k$  is congruent to 1 modulo  $u$ . Thus we have by a re-indexation in the sum

$$\chi(-t)\chi(k) \left( \sum_{m \in \mathbf{Z}^*} \frac{\chi^2(m)}{m^2} \right) = 2\chi(-t)\chi(k)L(2, \chi^2).$$

Finally, the constant term is zero unless  $\gcd(t, u) = 1$ . Since  $t|u$ , one obtains  $t = 1$ . The result follows from  $g(\bar{\chi}) = \chi(-1)g(\chi)$  and Lemma 3.  $\square$

*Proof.* (of Thm 1) Let us first prove the second part of the claim. If  $\chi$  is principal then  $\eta_\chi$  is an  $\eta$ -quotient by applying Lemma 2. We now assume the converse, i.e. that there exist a positive integer  $M$  and complex numbers  $c$  and  $(a_t)_{t|M}$  such that  $\eta_\chi = c \prod_{t|M} (\eta|V_t)^{a_t}$ . Writting  $\chi = \chi_0 \mathbf{1}_N$  where  $\chi_0$  is primitive and  $N$  is the modulus of  $\chi$ , thanks to Lemma 3 we obtain the equality:

$$\prod_{s|N} (\eta_{\chi_0}|V_s)^{\chi_0(s)\mu(s)} = c \prod_{t|M} (\eta|V_t)^{a_t}. \quad (2)$$

Taking the  $q$ -logarithmic derivative of Eq. (2), one obtains using Lemma 1:

$$\sum_{s|N} s\chi_0(s)\mu(s)g(\bar{\chi}_0) (E_2^{\chi_0, \bar{\chi}_0}|V_s) + \sum_{1 < t|M} a_t E_{2,t} - \left( \sum_{1 < t|M} a_t \right) E_2 = 0, \quad (3)$$

where we set  $E_{2,t} := E_2 - t(E_2|V_t)$ . As it is well-known, the elements  $E_2$ ,  $E_{2,t}$  ( $1 < t|NM$ ) and  $(E_2^{\varphi, \bar{\varphi}}|V_s)$  ( $\varphi$  primitive with modulus  $u$  and  $1 < su^2|NM$ ) form a basis of the space of Eisenstein series of weight 2 for  $\Gamma_0(NM)$  and thus are linearly independant. This implies the following: if  $\chi_0$  is non principal, then all the complex coefficients of Eq. (3) are equal to zero; it would imply that  $g(\bar{\chi}_0) = 0$  (taking the first coefficient of the first sum) which is a contradiction since  $\chi_0$  is primitive.

The function  $\eta_\chi$  as a cuspidal divisor: this is essentially Lemma 1. Let us show that  $\eta_\chi$  is a PGMF of weight 0 on  $\Gamma_0(u^2)$  when  $\chi$  is primitive with conductor  $u$ . As it is clear by now,  $\theta\eta_\chi/\eta_\chi$  is a classical modular form of weight 2 on  $\Gamma_0(u^2)$ . It remains to show condition (a) using Lemma 3. Let  $s$  be a cusp of  $\Gamma_0(u^2)$ : if  $s$  is not of the form  $k/u$  where  $k$  and  $u$  are coprime then the constant term of  $\theta\eta_\chi/\eta_\chi$  at the cusp  $s$ , namely  $l_s$ , is equal to zero. Otherwise,  $l_{k/u}$  is given by

$$w_{k/u} u^2 \chi(k) \left( \frac{u}{2\pi} \right)^2 L(2, \chi^2). \quad (4)$$

We shall show that the number given by (4) is an integer when  $\chi$  is real. In the latter case,  $\chi^2 = \mathbf{1}_u$  and the associated  $L$ -function at 2 is known to be  $\zeta(2) \prod_{p|u} (1 - 1/p^2)$ . The width  $w_{k/u}$  is equal to one so it yields after some simplifications:

$$\chi(k) \frac{u^4}{24} \prod_{p|u} \left( \frac{p^2 - 1}{p^2} \right). \quad (5)$$

Finally, it is sufficient to show that  $u^2 \prod_{p|u} (p^2 - 1)/24$  is an integer. Since  $u > 2$  (remember that there is no primitive character with modulus 2), then either 4 or  $p$  (where  $p \geq 3$  is a prime number) divide  $u$ . Since 24 divides  $p^2 - 1$  when  $p \geq 5$  is a prime number, a short calculation for  $3|u$  and  $4|u$  allows us to conclude that condition (a) is satisfied.

To prove that  $\eta_\chi$  is a PGMF when  $\chi$  with modulus  $N$  is not necessarily primitive, we make use of Lemma 2 to reduce to the case where  $\chi$  is primitive. We obtain that  $\eta_\chi$  is a PGMF on a certain group  $\Gamma_0(Q)$  where  $Q$  is the lcm of all the  $u^2 s$  ( $s|N$ ,  $u$  is the conductor of  $\chi$ ) taking in account that  $s$  is prime to  $u$  and square-free (i.e. that  $\chi_0(s)\mu(s)$  is non zero). We found  $Q = u^2 \prod_{p|N, p \nmid u} p$  and this finishes the proof.  $\square$

*Proof.* (of Thm 2) The Dirichlet characters  $(\cdot/3)$ ,  $\psi$  and  $(\cdot/5)$  are real and primitive. Applying Thm 1, we found that  $\eta_{(\cdot/3)}$ ,  $\eta_\psi$  and  $\eta_{(\cdot/5)}$  are PGMF of weight 0 with a cuspidal on the respective groups  $\Gamma_0(9)$ ,  $\Gamma_0(16)$  and  $\Gamma_0(25)$  which are not  $\eta$ -quotients. We finally make use of the remark in section 2: the multiplier systems of  $\eta_{(\cdot/3)}$ ,  $\eta_\psi$  and  $\eta_{(\cdot/5)}$  are unitary with order dividing 12.  $\square$

## IV Proof of Thm 3 and 4.

*Proof.* (of Thm 3 and 4) Let  $f$  be a modular function of integral weight  $k$  on  $\Gamma_0(N)$  with a cuspidal divisor. We can assume that  $k = 0$  taking  $f^{12}/\Delta$  (without loss of generality). Thanks to Knopp and Mason's result,  $\theta f/f$  is a classical modular form of weight 2 and thus the sum of an Eisenstein series and a cusp form. Denoting by  $g$  the cusp form part and writting the Eisenstein series in the basis formed by  $E_2$ ,  $E_{2,t}$  ( $1 < t|N$ ) and  $(E_2^{\varphi, \bar{\varphi}}|V_s)$  ( $\varphi$  primitive with modulus  $u$  and  $1 < su^2|N$ ), it gives (using Lemma 1):

$$\frac{\theta f}{f} = g + \sum_{\chi, t} ta_{t, \chi} (\theta \eta_\chi / \eta_\chi | V_t)$$

for some complex numbers  $(a_{t, \chi})$ , where the sum is indexed by integers  $t$  and primitive characters  $\chi$  of conductor  $u$  such that  $1 < tu^2|N$ . Finally, one finds that  $f = G \prod_{t, \chi} (\eta_\chi | V_t)^{a_{t, \chi}}$  where  $G$  is a function on  $\mathcal{H}$  satisfying  $\theta G/G = g$  and thus a PGMF (whose  $q$ -logarithmic derivative is a cusp form) thanks to Knopp and Mason's theorem.

**If  $N = 4M$  or  $8M$  where  $M$  is odd and square-free:**  $u^2|N$ , thus  $u$  is either 1 or 2 and since there is no primitive character with conductor 2,  $u = 1$  and  $\chi$  is principal. This yields

$$G = f^{-1} \prod_{1 < t|N} (\eta | V_t)^{a_{t, 1_1}}$$

which gives that  $G$  has unitary character:  $G$  is thus constant by the argument given in section 2.  $f$  is an  $\eta$ -quotient and this proves Thm 3.

**If  $N = 9M$  or  $27M$  where  $M$  is square-free and coprime with 3:** since  $u^2|N$ ,  $u$  is either 1 or 3 and  $\chi$  is either  $1_1$  or  $(\cdot/3)$ ;

**If  $N = 16M$  or  $32M$  where  $M$  is odd and square-free:** since  $u^2|N$ ,  $u$  is either 1 or 4 and  $\chi$  is either  $1_1$  or  $\psi$ ;

**If  $N = 25M$  or  $125M$  where  $M$  is square-free and coprime with 5:** since  $u^2|N$ ,  $u$  is either 1 or 5 and  $\chi$  is either  $1_1$ ,  $(\cdot/5)$ ,  $\chi_1$  or  $\chi_2$ ; it is now easy to mimic the precedent proof of Thm 3 and finish the proof of Thm 4 (using Thm 2).  $\square$

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